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Quaternionic time-harmonic Maxwell operator

Vladislav V Kravchenko† and Michael V Shapiro‡

† Section of Informatics, Escuela Superior de Ingenieria y Mecanica Electrica del IPN, Unidad-Zacatenco, Avenida Instituto Politecnico Nacional S/N, Edificio 'ISLA', CP 07738, Mexico, DF, Mexico

‡ Departamento de Matemáticas, Escuela Superior de Fisica y Matematicas del IPN, 07300 Mexico DF, Mexico

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Abstract. The theory of quaternionic α-hyperholomorphic functions (synonyms: monogenic, regular, spatial holomorphic vectors) is currently developing rapidly. In particular, many integral formulae with explicit reproducing kernels have been obtained. In this work we establish a one-to-one correspondence between time-harmonic (= monochromatic) electromagnetic fields and pairs of 'mutually conjugate' hyperholomorphic functions. This leads to the Cauchy-type integral associated with Maxwell's equations. Some main integral formulae for Maxwell's equations involving this Cauchy-type integral are obtained. It should be mentioned that, in fact, we introduce and study a somewhat more general quaternionic object which has better algebraic and analytic properties than the 'physical' Maxwell operator and which contains the latter as a special case.

0. Introduction

Since J C Maxwell wrote and published his famous equations, they have been investigated in a large number of works. There probably exists no fewer works generalizing these equations in many diverse directions. There is no need to spend many words explaining the reasons for such phenomena, they are evident: the importance of the subject. At the same time, the necessity of studying the equation for more than a century bears witness to the absence of a sufficiently complete theory for Maxwell's equations.

In the present work we make an attempt to construct a function theory associated with the monochromatic (in the literature the synonym 'time-harmonic' is often also used) Maxwell equations with constant coefficients, in the framework of exploiting hypercomplex function theory.

Various hypercomplex approaches to studying the classical Maxwell equations have more than a century of history, starting from the work of Maxwell himself (which sometimes surprises both mathematicians and physicists). There exists a well known reformulation of these equations in vacuum in quaternionic terms (see e.g. [34, 5, 16, 30, 17, 1, 19]), which allows some fundamental physical laws to be rewritten in a space-saving form. This is the very case in which such a phenomenological simplification is a real discovery influencing the development of a physical theory.

Formally, this leads to a partial differential operator with quaternionic coefficients which has a null-set containing all solutions to the Maxwell equations. So the problem arises as to whether this null-set possesses a well-developed function theory. The latter means a deep structural analogy with one-dimensional complex analysis which provides, first of all, for the existence of an integral representation for the null-solutions with a good analogue of the complex Cauchy kernel. Only in this case can we expect to get a theory (almost) as rich as the theory of holomorphic functions of one complex variable.

For many specific radio engineering, hydroacoustical and geophysical models it is natural and quite sufficient to limit the study to the time-harmonic case (see e.g. [14, 28, 6, 18, 10, 15, 7] and many other books and articles). The main reason is contained, in fact, in the Fourier analysis together with the principle of superposition: an electromagnetic wave is a superposition (or, in other words, a linear combination, finite or numerable) of elementary, periodic-in-time waves. Of course, technically the time-harmonic case is simpler but at the same time, even for that case, many profound physical properties are not understood and explained. For example: the behaviour of the electromagnetic vector field near and on the boundary of a spatial domain until now has been far from having a sufficiently complete description. Most of what is known is contained in [6], see also the 'less rigorously mathematical' book [36] where many interesting results and ideas can be found. It is written in traditional vectorial language (as is [6]) but there are some important hints as to how to develop the corresponding hypercomplex approach.

The main difference between our work and that mentioned above [34, 16, 17, 1] and others in this direction, consists of the following. We not only rewrite the Maxwell equations in a space-saving form (which generally speaking would not give essentially new information) but also with the aid of a simple matrix transform we imbed the time-harmonic electromagnetic field theory, which is difficult to treat, into the sufficiently developed [29, 12, 13, 27, 35, 4, 31, 23–25, 21, 26] theory of α-hyperholomorphic biquaternionic functions. This allows new facilities for solving the boundary value problems arising in electrodynamics because, following the deep structural analogy with one-dimensional complex analysis and having the Cauchy-type operator associated with Maxwell's equations (see section 3), one is able to analyse and to solve the analogues of the Riemann and Hilbert problems as well as some inverse problems [11] for electromagnetic fields. For some results in this direction we refer the reader to [26].

Let $\mathbb{H}(\mathbb{C})$ denote the algebra of complex quaternions (precise definitions and some properties are given in section 1), and let Ω be a domain in $\mathbb{R}^3$.

In a series of our works [23–26] (see also [21]) we have constructed a function theory for the null-solutions of the operator

$$ {}^{\psi}D_\alpha := \sum_{k=1}^{3} \psi_k \cdot \frac{\partial}{\partial x_k} + M^\alpha \tag{0.1} $$

where $\alpha \in \mathbb{H}(\mathbb{C})$, M^α denotes an operator of multiplication by α on the right, and the set $\psi := \{\psi^1, \psi^2, \psi^3\}$ is chosen in such a way that ensures a factorization of the Helmholtz operator: if

$$ {}^{\bar{\psi}}D_\alpha := M^\alpha - \sum_{k=1}^{3} \psi_k \cdot \frac{\partial}{\partial x_k} \tag{0.2} $$

then

$$ {}^{\psi}D_\alpha \cdot {}^{\bar{\psi}}D_\alpha = {}^{\bar{\psi}}D_\alpha \cdot {}^{\psi}D_\alpha = M^{\alpha^2} + \Delta_{\mathbb{R}^3} \tag{0.3} $$

$\Delta_{\mathbb{R}}^3$ being the Laplacian in $\mathbb{R}^3$.

It appears that this function theory alone (we call it a (ψ, α)-hyperholomorphic function theory) possesses all the necessary peculiarities of one-dimensional complex analysis. Moreover, as was shown in [23, 20, 21, 26], each component of the electromagnetic vector field $(\boldsymbol{E}, \boldsymbol{H})$ is a linear combination of two purely imaginary quaternionic functions: one of which is a null-solution of (0.1); and the other a null-solution of its conjugate (0.2), for some specific value of α.

Thus we have reduced a theory of monochromatic solutions of the Maxwell equations to hyperholomorphic function theory which conserves all the essential properties of one-dimensional complex analysis, and so these properties can be obtained for monochromatic waves.

It is important to mention that, in fact, we treat a somewhat more general quaternionic object (for obvious reasons we call it the quaternionic Maxwell operator). It has better algebraic and analytic properties compared with the 'physical' Maxwell operator, and it contains the latter as a special case.

Besides the introduction, the work consists of three sections. Section 1 contains a very brief description of the common Maxwell equations. Some necessary functional spaces are also introduced. Section 2 contains the necessary mathematical tools. The definition and basic properties of complex quaternions are given. Then we explain what an α-hyperholomorphic function is together with its main properties. Let us stress that for $\alpha \neq 0$ all this is quite new, and it is almost unknown both among mathematicians and physicists. Just for this reason we put it in a special section.

Section 3 is central. First of all we establish an explicit connection between the Maxwell operator and a pair of 'mutually conjugate' quaternionic operators of the Cauchy–Riemann type which generate the exact analogues of the usual holomorphic (= analytic) functions of one complex variable. Then in theorem 5 the null-set of the three-dimensional Helmholtz operator is decomposed into the direct sum of two functional spaces each of which is a 'rotated' null-set of the quaternionic Maxwell operator. In theorem 7 the variants of the Cauchy integral theorem for the Maxwell functions are formulated. Then we introduce the analogues of the Cauchy-type operator and T-operator corresponding to Maxwell's equations. Finally we give three theorems which show how the introduced operators work.

1. Maxwell's equations

Let Ω denote a domain in $\mathbb{R}^3$, and let $\boldsymbol{E}$ and $\boldsymbol{H}$ be the corresponding electrical and magnetic components of an electromagnetic field in Ω. We assume that a medium in Ω is homogeneous and that there are no currents and charges in Ω. If an electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ is time-harmonic (or monochromatic which is a synonym) then it satisfies the following Maxwell equations:

$$\operatorname{rot} \boldsymbol{H} = \sigma \boldsymbol{E} \qquad \operatorname{rot} \boldsymbol{E} = \mathrm{i}\omega\mu \boldsymbol{H} \tag{1.1}$$

$$\operatorname{div} \boldsymbol{H} = 0 \qquad \operatorname{div} \boldsymbol{E} = 0 \tag{1.2}$$

where $\sigma := \sigma^* - \mathrm{i}\omega\epsilon$ is the complex electrical conductivity; ϵ is the dielectric constant; μ is the magnetic permeability; σ^* is the medium electrical conductivity which is inverse to its electrical resistivity: $\sigma^* = \rho^{-1}$. It is known also that in this situation the complex vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the homogeneous Helmholtz equations:

$$\Delta \boldsymbol{E} + \lambda \boldsymbol{E} = 0 \tag{1.3}$$

$$\Delta \boldsymbol{H} + \lambda \boldsymbol{H} = 0 \tag{1.4}$$

where $\lambda := \mathrm{i}\omega\mu\sigma^* + \omega^2\mu\epsilon = \mathrm{i}\omega\mu\sigma \in \mathbb{C}$, $\alpha := \sqrt{\lambda}(\operatorname{Re}\alpha > 0)$ is a medium wavenumber.

For any vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ by the definitions

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle := \sum_{k=1}^{3} f_k g_k \in \mathbb{C} \tag{1.5}$$

$$[f, g] := \begin{vmatrix} i_1 & i_2 & i_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} \in \mathbb{C}^3. \tag{1.6}$$

(1.5) and (1.6) are called, as in the case of real coordinates, the scalar and vector (= cross) product respectively. Equality (1.5) defines a $\mathbb{C}$-valued bilinear form over $\mathbb{C}$, the equality (1.6) defines a $\mathbb{C}^3$-valued bilinear form over $\mathbb{C}$.

Remark. It is easy to see that the equalities (1.2) are deduced directly from (1.1), and so there is no need to include them in the Maxwell system. However, in many physical sources, books and articles, the equations (1.2) are included explicitly, and we follow this tradition here.

Let us remember that any solution of the system (1.1)–(1.2) is 'orthogonal' with respect to the bilinear form (1.5):

$$\langle \boldsymbol{E}, \boldsymbol{H} \rangle = 0 \tag{1.7}$$

(see the proof, e.g., in [15]).

Equation (1.1) can be rewritten in the matrix–vector form

$$\begin{pmatrix} \sigma & -\mathrm{rot} \\ \mathrm{rot} & -\mathrm{i}\omega\mu \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = 0 \tag{1.8}$$

and hence we can consider the operator defined by the matrix on the left-hand side of (1.8):

$$\mathcal{M} := \begin{pmatrix} \sigma & -\mathrm{rot} \\ \mathrm{rot} & -\mathrm{i}\omega\mu \end{pmatrix}. \tag{1.9}$$

Its natural domain is $C^1(\Omega; \mathbb{C}^3 \times \mathbb{C}^3)$. Taking into account (1.2) let us introduce for $k \in \mathbb{N} \cup \{0\}$

$$\hat{C}^k := \hat{C}^k(\Omega; \mathbb{C}^3 \times \mathbb{C}^3) := \{(f, g) | (f, g) \in \mathbb{C}^3 \times \mathbb{C}^3; \operatorname{div} f = \operatorname{div} g = 0\}. \tag{1.10}$$

We will call the operator

$$\hat{\mathcal{M}} := \mathcal{M}|\hat{C}^1 \tag{1.11}$$

the time-harmonic Maxwell operator. It is essential to note that $\hat{\mathcal{M}}$ maps a solenoidal (= divergenceless) vector into a solenoidal one reducing the smoothness: if $(f, g) \in \hat{C}^2$, $(\boldsymbol{u}, \boldsymbol{v}) := \hat{\mathcal{M}}[(f, g)]$ then $(\boldsymbol{u}, \boldsymbol{v}) \in \hat{C}^1$.

2. Basic facts of hyperholomorphic function theory

Let $\mathbb{H}$ be a set of the real quaternions. This means that elements of $\mathbb{H}$ are of the form $a = \sum_{k=0}^{3} a_k i_k$, where $\{a_k | k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup [0]; \mathbb{N}_3 := \{1, 2, 3\}\} \subset \mathbb{R}$; i_0 is the unit; i_1, i_2, i_3 are called the imaginary units, and they define arithmetic rules in $\mathbb{H}$: by definition $i_k^2 = -i_0$, $k \in \mathbb{N}_3$; $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$.

Natural operations of addition and multiplication in $\mathbb{H}$ turn it into a non-commutative field (= a skew field). There is a main involution in $\mathbb{H}$ called the quaternionic conjugation, and it plays an exceptionally significant role. This involution is defined in the following way:

$$\bar{i}_0 := i_0 \qquad \bar{i}_k = -i_k \qquad k \in \mathbb{N}_3$$

and it extends onto $\mathbb{H}$ by $\mathbb{R}$-linearity, that is, for $a \in \mathbb{H}$

$$\bar{a} := \sum_{k=0}^{3} \overline{a_k i_k} = \sum_{k=0}^{3} a_k \bar{i}_k = a_0 - \sum_{k=1}^{3} a_k i_k =: Z(a). \tag{2.1}$$

Note that

$$\bar{a} \cdot a = a \cdot \bar{a} = \sum_{k=0}^{3} a_k^2 = |a|^2_{\mathbb{R}^4} =: |a|^2_{\mathbb{H}}. \tag{2.2}$$

Therefore for $a \in \mathbb{H}\setminus\{0\}$ a quaternion $a^{-1} := \frac{1}{a\bar{a}}\bar{a} = \frac{1}{|a|^2}\bar{a}$ is an inverse to a. We should mention also the very important property of the quaternionic conjugation: for $\forall\{a, b\} \subset \mathbb{H}$

$$\overline{a \cdot b} = \bar{b} \cdot \bar{a} \tag{2.3}$$

or

$$Z(a \cdot b) = Z(b) \cdot Z(a). \tag{2.4}$$

For our purposes we need the notion of a complex quaternion. The set of complex quaternions $\mathbb{H}(\mathbb{C})$ consists of the elements $a = \sum_{k=0}^{3} a_k \cdot i_k$ where $\{i_k\}$ is as described above, $\{a_k\} \subset \mathbb{C}$, the set of usual complex numbers with the imaginary unit i. By definition

$$i \cdot i_k = i_k \cdot i \qquad k \in \mathbb{N}_3^0.$$

Arithmetic rules are defined in $\mathbb{H}(\mathbb{C})$ just as in $\mathbb{H}$. It is obvious that $\mathbb{H}$ is a real subalgebra in $\mathbb{H}(\mathbb{C})$. Each element $a \in \mathbb{H}(\mathbb{C})$ can be represented in the form

$$a = a^{(1)} + i \cdot a^{(2)} = a^{(1)} + a^{(2)} \cdot i \tag{2.5}$$

where $\{a^{(1)}, a^{(2)}\} \subset \mathbb{H}$.

Conjugation acts only on the quaternionic units, not on i, in (2.1). The properties (2.3)–(2.4) remain true but instead of (2.2) we have

$$\begin{aligned} a\bar{a} = \bar{a}a &= \sum_{k=0}^{3} a_k^2 = |a^{(1)}|^2 - |a^{(2)}|^2 + i(a^{(1)}\overline{a^{(2)}} + a^{(2)}\overline{a^{(1)}}) \\ &= |a^{(1)}|^2 - |a^{(2)}|^2 + 2i\langle a^{(1)}, a^{(2)}\rangle \in \mathbb{C} \end{aligned} \tag{2.6}$$

where $|a^{(k)}|$ stands for the usual module of a real quaternion (see (2.2)), $\langle a^{(1)}, a^{(2)}\rangle$ a scalar product of two four-dimensional vectors. (2.6) means that

$$a \cdot \bar{a} \neq |a|^2_{\mathbb{R}^8} := \sum_{k=0}^{3} |a_k|^2 = |a^{(1)}|^2 + |a^{(2)}|^2$$

and that $\mathbb{H}(\mathbb{C})$ has zero divisors. The set of all zero divisors we denote by $\mathfrak{S}$, i.e.

$$\mathfrak{S} := \{a \in \mathbb{H}(\mathbb{C}) | a \neq 0; \exists b \neq 0 \colon ab = 0\}.$$

Let $G\mathbb{H}(\mathbb{C})$ denotes the group of invertible elements from $\mathbb{H}(\mathbb{C})$: $G\mathbb{H}(\mathbb{C}) := \mathbb{H}(\mathbb{C})\setminus\mathfrak{S}\cup\{0\}$. Then for any $a \in G\mathbb{H}(\mathbb{C})$ the quaternion $a^{-1} := \bar{a}/(a\bar{a})$ is its inverse.

From representation (2.5) it clearly follows that the set of complex quaternions is isomorphic as a real vectorial space to the set of octonions (Cayley numbers). So the difference between the sets lies on the algebraic level. By the definition of octonions the additional imaginary unit i anticommutes with i_k, $k \in \mathbb{N}_3$ and, as a consequence, the algebra of octonions does not enjoy the property of associativity against the algebra of complex quaternions (see, e.g., [32, 9]).

Denoting for $a \in \mathbb{H}$ or $a \in \mathbb{H}(\mathbb{C})$

$$a_0 =: \mathrm{Sc}(a) \qquad \boldsymbol{a} := \sum_{k=1}^{3} a_k \cdot i_k =: \mathrm{Vect}(a)$$

we can write

$$a = a_0 + \boldsymbol{a}.$$

a_0 is called the scalar part of a quaternion a, $\boldsymbol{a}$ is called the vector part, or the purely imaginary part. Using the notation (1.5)–(1.6) we have, for any complex quaternions a and b,

$$\begin{aligned} a \cdot b &= (a_0 + \boldsymbol{a}) \cdot (b_0 + \boldsymbol{b}) \\ &= a_0 \cdot b_0 - \langle \boldsymbol{a}, \boldsymbol{b} \rangle + a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + [\boldsymbol{a}, \boldsymbol{b}]. \end{aligned} \tag{2.7}$$

This equality takes the most impressive form if $a_0 = b_0 = 0$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = -\langle \boldsymbol{a}, \boldsymbol{b} \rangle + [\boldsymbol{a}, \boldsymbol{b}]. \tag{2.8}$$

Equality (2.8) contains three types of multiplication of three-dimensional vectors, and it can be used to imply and to explain all the rules of scalar and vector products.

We shall consider $\mathbb{H}(\mathbb{C})$-valued functions defined in Ω:

$$f : \Omega \to \mathbb{H}(\mathbb{C}).$$

The notation $C^p(\Omega; \mathbb{H}(\mathbb{C}))$, $p \in \mathbb{N} \cup \{0\}$, has the usual component-wise meaning. If a is a fixed $\mathbb{H}(\mathbb{C})$-valued function, then M^a and ${}^aM$ are the operators defined on a set of functions $\{f\}$ by the rule

$$M^a[f] := fa \qquad {}^aM[f] := af.$$

Let $\psi := \{\psi^1, \psi^2, \psi^3\} \subset \mathbb{H}$ and $\mathrm{Sc}(\psi^1) = \mathrm{Sc}(\psi^2) = \mathrm{Sc}(\psi^3) = 0$. Thus ψ is a set of three-dimensional vectors. Assume, in addition, that ψ is an orthonormalized system. Sometimes we shall call ψ a structural set. On a set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ define the operators ${}^\psi D$ and D^ψ by the equalities

$${}^\psi D[f] := \sum_{k=1}^{3} \psi^k \cdot \frac{\partial f}{\partial x_k} =: \sum_{k=1}^{3} \psi^k \cdot \partial_k[f] \tag{2.9}$$

$$D^\psi[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial x_k} \cdot \psi^k =: \sum_{k=1}^{3} \partial_k[f] \cdot \psi^k. \tag{2.10}$$

Let Δ be the three-dimensional Laplace operator: $\Delta = \sum_{k=1}^{3} \partial_k^2$. Define Δ on the $\mathbb{H}(\mathbb{C})$-valued functions by the equality

$$\Delta[f] := \sum_{k=0}^{3} \Delta[f_k] i_k.$$

Then on $C^2(\Omega; \mathbb{H}(\mathbb{C}))$ the following equalities are true:

$$\Delta = {}^\psi D \, {}^{\bar\psi} D = -{}^\psi D \, {}^\psi D = D^\psi D^{\bar\psi} = -D^\psi D^\psi. \tag{2.11}$$

For details see, for example, [33].

Let $\lambda \in \mathbb{C}\backslash\{0\}$ and let α denote an arbitrary fixed square root of λ in $\mathbb{H}(\mathbb{C})$, i.e. a solution in $\mathbb{H}(\mathbb{C})$ of the equation $\alpha^2 = \lambda$. This λ generates the three-dimensional Helmholtz operator

$$\Delta_\lambda := \Delta + \lambda \tag{2.12}$$

which acts on $C^2(\Omega; \mathbb{H}(\mathbb{C}))$. For an arbitrary fixed set ψ and for the above stated α let us introduce the operators

$${}^\psi D_\alpha := M^\alpha + {}^\psi D \qquad {}^{\bar\psi} D_\alpha := M^\alpha - {}^\psi D \tag{2.13}$$

$${}_\alpha D^\psi := {}^\alpha M + D^\psi \qquad {}_\alpha D^{\bar\psi} := {}^\alpha M - D^\psi. \tag{2.14}$$

We shall call the operators (2.13) the left mutually conjugate Cauchy–Riemann operators and, for the operators (2.14), the word 'left' changes for 'right'. From (2.11) it follows that the following equalities are true:

$$\Delta_\lambda = {}^{\psi}D_\alpha \cdot {}^{\bar\psi}D_\alpha = {}^{\bar\psi}D_\alpha \cdot {}^{\psi}D_\alpha = {}_\alpha D^{\psi} \cdot {}_\alpha D^{\bar\psi} = {}_\alpha D^{\bar\psi} \cdot {}_\alpha D^{\psi}. \tag{2.15}$$

They give the factorization of the Helmholtz operators with a complex parameter λ. Thus a family (depending on the parameters ψ and α) of the operators (2.13) and (2.14), factorizing the Helmholtz operator, is obtained.

Let us fix ψ, α, Ω and introduce the sets

$${}^{\psi}\mathfrak{M}_\alpha(\Omega; \mathbb{H}(\mathbb{C})) := \ker {}^{\psi}D_\alpha \qquad {}_\alpha\mathfrak{M}^{\psi}(\Omega; \mathbb{H}(\mathbb{C})) := \ker {}_\alpha D^{\psi}. \tag{2.16}$$

Sometimes, elements of these sets will be referred to as left- (correspondingly right-) (ψ, α)-hyperholomorphic in $\Omega\,\mathbb{H}(\mathbb{C})$-valued functions; this name will be shortened if misunderstanding cannot arise. The sets (2.16) will be called the classes of (ψ, α)-hyperholomorphy shortly. Any such left (correspondingly right) class forms a right (correspondingly left) $\{c\alpha | c \in \mathbb{C}\}$-submodule of an $\mathbb{H}(\mathbb{C})$-module $C^1(\Omega; \mathbb{H}(\mathbb{C}))$. It is clear that for some special values of α this fact could be specified.

The equalities (2.15) mean that (ψ, α)-hyperholomorphic functions of a class C^2 are metaharmonic in Ω, that is they belong to $\ker \Delta_\lambda$.

Investigation of the time-harmonic Maxwell operator requires us to consider, as a rule, the case $\psi = \psi_{\mathrm{st}} = \{i_1, i_2, i_3\}$, for which we will use notation

$$\begin{aligned} {}^{\psi_{\mathrm{st}}}D_\alpha &=: {}^{\mathrm{st}}D_\alpha =: D_\alpha \qquad & {}_\alpha D^{\psi_{\mathrm{st}}} &=: {}_\alpha D^{\mathrm{st}} =: {}_\alpha D \\ {}^{\overline{\psi_{\mathrm{st}}}}D_\alpha &=: {}^{\overline{\mathrm{st}}}D_\alpha =: \bar D_\alpha \qquad & {}_\alpha D^{\overline{\psi_{\mathrm{st}}}} &=: {}_\alpha D^{\overline{\mathrm{st}}} =: {}_\alpha \bar D. \end{aligned} \tag{2.17}$$

For this special case the vector representation of a quaternion (see earlier) gives rise to the following representation of the operators ${}^{\mathrm{st}}D$ and ${}^{\mathrm{st}}D_\alpha$: for $\forall f \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$

$$\begin{aligned} D[f] &= \sum_{k=1}^{3} i_k \frac{\partial f}{\partial x_k} = \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}(f_0 + \vec f) \\ &= \sum_{k=1}^{3} i_k \frac{\partial f_0}{\partial x_k} - \sum_{k=1}^{3} \left\langle i_k, \frac{\partial \vec f}{\partial x_k} \right\rangle + \sum_{k=1}^{3} \left[i_k, \frac{\partial \vec f}{\partial x_k} \right] \\ &= \operatorname{grad} f_0 - \operatorname{div} \vec f + \operatorname{rot} \vec f. \end{aligned} \tag{2.18}$$

Hence

$${}^{\mathrm{st}}D_\alpha[f_0 + \vec f] = -\operatorname{div} \vec f - \langle \vec f, \vec\alpha \rangle + \alpha_0 f_0 + \operatorname{grad} f_0 + \operatorname{rot} \vec f + [\vec f, \vec\alpha] + f_0 \cdot \vec\alpha + \alpha_0 \cdot \vec f. \tag{2.19}$$

In what follows let $0 \neq \lambda \in \mathbb{C}$, $\alpha \in \mathbb{C}$.

Theorem 1. [23]. Let $\mathcal{H}_\lambda(\Omega, \mathbb{H}(\mathbb{C})) := \ker \Delta_\lambda$ be the space of metaharmonic functions; denote ${}^{\psi}\pi_\alpha := \frac{1}{2\alpha}\,{}^{\psi}D_\alpha$, ${}^{\psi}\Pi_\alpha := {}^{\psi}\pi_\alpha|\mathcal{H}_\lambda$.

Then

(i) The following correlations are true:

(a) ${}^{\psi}\Pi_\alpha^2 = {}^{\psi}\Pi_\alpha$,

(b) ${}^{\psi}\Pi_\alpha \cdot {}^{\bar\psi}\Pi_\alpha = {}^{\bar\psi}\Pi_\alpha \cdot {}^{\psi}\Pi_\alpha = 0$,

(c) ${}^{\psi}\Pi_\alpha + {}^{\bar\psi}\Pi_\alpha = I$, the identity operator,

(d) $\rangle({}^{\psi}\Pi_\alpha) = \ker({}^{\bar\psi}\pi_\alpha|C^2)$;

(ii) For ${}^{\psi}\mathcal{H}_\alpha := {}^{\psi}\Pi_\alpha(\mathcal{H}_\lambda)$, $\mathcal{H}_\lambda = {}^{\psi}\mathcal{H}_\alpha \oplus {}^{\bar\psi}\mathcal{H}_\alpha$;

(iii) ${}^{\psi}\mathcal{H}_\alpha = \ker({}^{\bar\psi}\pi_\alpha|C^2)$;

(iv) $\mathcal{H}_\lambda = {}^{\psi}\mathfrak{M}_\alpha \oplus {}^{\bar\psi}\mathfrak{M}_\alpha$.

Theorem 2. (The Stokes formula congruent with the notion of (ψ, α)-hyperholomorphy.) Let $\{f, g\} \subset C^1(\bar{\Omega}; \mathbb{H}(\mathbb{C}))$. Then

(i)

$$d(g \cdot \sigma_{\psi,x} \cdot f) = \begin{cases} ({}_\alpha D^\psi[g] \cdot f + g \cdot {}^\psi D_\alpha[f] - 2\alpha \cdot g \cdot f)\,\mathrm{d}x \\ ({}_\alpha D^\psi[g] f - g \cdot {}^{\bar\psi} D_\alpha[f])\,\mathrm{d}x \end{cases}$$

(ii)

$$\int_\Gamma g(x) \cdot \sigma_{\psi,x} \cdot f(x) = \begin{cases} \int_\Omega ({}_\alpha D^\psi[g](x) \cdot f(x) + g(x) \cdot {}^\psi D_\alpha[f](x) - 2\alpha g(x) f(x))\,\mathrm{d}x, \\ \int_\Omega ({}_\alpha D^\psi[g] \cdot f - g \cdot {}^{\bar\psi} D_\alpha[f])\,\mathrm{d}x \end{cases}$$

where $\sigma_{\psi,x} := \sum_{k=1}^{3}(-1)^{k-1}\psi^k\,\mathrm{d}\hat{x}^k$ and $\mathrm{d}\hat{x}^k$ denotes, as usual, the differential form $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ with the factor $\mathrm{d}x^k$ omitted. (Note that if $\mathrm{d}\Gamma$ is an element of the surface area in $\mathbb{R}^3$ then $|\sigma_{\psi,x}| = \mathrm{d}\Gamma$ and if Γ is a smooth surface, then

$$\sigma_{\psi,x} = \sum_{k=1}^{3} \psi^k n_k(x)\,\mathrm{d}\Gamma$$

where $n := (n_1, n_2, n_3)$ is a unit vector of the outward normal on Γ at the point $x \in \Gamma$.)

Theorem 3. (Variants of the Cauchy integral theorem for (ψ, α)-hyperholomorphic functions.)

(i) Let $f \in {}^\psi\mathfrak{M}_\alpha(\bar{\Omega}; \mathbb{H}(\mathbb{C}))$, $g \in {}_\alpha\mathfrak{M}^\psi(\bar{\Omega}; \mathbb{H}(\mathbb{C}))$. Then

$$d(g \cdot \sigma_{\psi,x} \cdot f) = -2\alpha \cdot g \cdot f$$
$$d(\sigma_{\psi,x} \cdot f) = -\alpha f\,\mathrm{d}x$$
$$d(g \cdot \sigma_{\psi,x}) = -\alpha g\,\mathrm{d}x.$$

(ii) For the same functions

$$\int_\Gamma g(x) \cdot \sigma_{\psi,x} \cdot f(x) = -2\alpha \int_\Omega g(x) \cdot f(x)\,\mathrm{d}x$$
$$\int_\Gamma \sigma_{\psi,x} \cdot f(x) = -\alpha \int_\Omega f(x)\,\mathrm{d}x$$
$$\int_\Gamma g(x) \cdot \sigma_{\psi,x} = -\alpha \int_\Omega g(x)\,\mathrm{d}x.$$

(iii) If $f \in {}^\psi\mathfrak{M}_\alpha$, $g \in {}_\alpha\mathfrak{M}^{\bar\psi}$ then

$$\int_\Gamma g(x) \cdot \sigma_{\psi,x} \cdot f(x) = 0.$$

Let $\theta_\lambda^\pm$ denote the fundamental solution of the Helmholtz operator:

$$\theta_\lambda^\pm(x) = -\frac{1}{4\pi|x|}\mathrm{e}^{\pm\mathrm{i}\alpha|x|} \qquad x \in \mathbb{R}^3 \backslash \{0\} \tag{2.20}$$

where $\alpha^2 = \lambda$, $\mathrm{Re}\,\alpha > 0$. In particular, for $\alpha = 0$ we obtain $\theta_0 := \theta_0^\pm$, the fundamental solution of the Laplace operator. For the calculations, the following correlation which can be easily obtained is useful:

$$\theta_\lambda^\pm(x) = \theta_0(x) \mp \mathrm{i}\alpha\frac{1}{4\pi} + o(|x|)$$

when $|x| \to 0$. Below $\theta_\lambda := \theta_\lambda^-$ participates only. For θ_λ^+ all can be done by analogy.

Now the fundamental solution (= the Cauchy kernel for the corresponding theory) of the operator ${}^\psi D_\alpha$ is given by the formula

$$\mathcal{K}_{\psi,\alpha}(x) := {}_\alpha D^{\bar\psi}[\theta_\lambda](x) = {}^{\bar\psi} D_\alpha[\theta_\lambda](x)$$
$$= \frac{1}{|x|^2} \cdot \left(\alpha \cdot |x|^2 + (i\alpha|x| + 1) \cdot \sum_{k=1}^{3} \psi^k \cdot x_k \right) \cdot \theta_\lambda(x). \tag{2.21}$$

This Cauchy kernel generates, as usual, two important integrals:

$${}^\psi K_\alpha[f](x) := -\int_\Gamma \mathcal{K}_{\psi,\alpha}(x-\tau) \cdot \sigma_{\psi,\tau} \cdot f(\tau) \tag{2.22}$$

and

$${}^\psi T_\alpha[f](x) := \int_\Omega \mathcal{K}_{\psi,\alpha}(x-\tau) f(\tau)\, \mathrm{d}\tau. \tag{2.23}$$

The following propositions hold.

Proposition 1. (The Borel–Pompeiu (= Cauchy–Green) formula for (ψ,α)-h.h.f. theory.) *Let Ω be a bounded domain in $\mathbb{R}^3$ with a Liapunov boundary $\Gamma = \partial\Omega$; $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\bar\Omega; \mathbb{H}(\mathbb{C}))$.* Then

$$f(x) = {}^\psi K_\alpha[f](x) + {}^\psi T_\alpha \cdot {}^\psi D_\alpha[f](x) \qquad \text{for } \forall x \in \Omega. \tag{2.24}$$

Proposition 2. Let $f \in C^{0,\mu}(\Omega) \cap C(\bar\Omega), 0 < \mu \leqslant 1$ then in Ω

$${}^\psi D_\alpha \cdot {}^\psi T_\alpha[f](x) = f(x). \tag{2.25}$$

Proposition 3. (Cauchy integral formula.) If f satisfies the condition of the Borel–Pompeiu formula and $f \in {}^\psi\mathfrak{M}_\alpha(\Omega)$ then

$$f(x) = {}^\psi K_\alpha[f](x) \qquad \text{for } \forall x \in \Omega. \tag{2.26}$$

Remark. For the case $\alpha = 0$ the above theorems have been known for a long time (see, e.g., [3] as well as [8] for the Clifford algebra-valued functions).

3. Relationship between the time-harmonic Maxwell operator and the quaternionic Cauchy–Riemann operators

There exists an intimate connection between the time-harmonic Maxwell operator $\hat{\mathcal{M}}$ and the Cauchy–Riemann operators D_α, $\bar D_\alpha$,—and hence between the time-harmonic electromagnetic fields and hyperholomorphic functions. We now start to discuss this connection.

Equality (2.18) means, in particular, that on the solenoidal vector fields D acts as the operator rot: if $f_0 = 0_\Omega$, $\operatorname{div} \boldsymbol{f} = 0$ then

$$D[\boldsymbol{f}] = \operatorname{rot} \boldsymbol{f}. \tag{3.1}$$

This pushes us to introduce the following matrix operator (compare with (1.9)):

$$\mathcal{N} := \begin{pmatrix} \sigma & -D \\ D & -\mathrm{i}\omega\mu \end{pmatrix} : C^1(\Omega; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C})) \to C^0(\Omega; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C})). \tag{3.2}$$

We will call it the quaternionic Maxwell operator. Its restriction $\tilde{\mathcal{N}}$ onto $\mathbb{C}^3 \times \mathbb{C}^3$-valued functions has the form

$$\tilde{\mathcal{N}} = \begin{pmatrix} \sigma; & \operatorname{div} - \operatorname{rot} \\ -\operatorname{div} + \operatorname{rot}; & -\mathrm{i}\omega\mu \end{pmatrix}$$

and hence does not coincide with $\mathcal{M}$ (moreover, $\tilde{\mathcal{N}}$ maps such functions onto $\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C})$-valued functions). But the restriction $\mathcal{N}|\hat{C}^1$ gives just $\hat{\mathcal{M}}$:

$$\hat{\mathcal{M}} = \mathcal{N}|\hat{C}^1. \tag{3.3}$$

In particular, this justifies, in our opinion, the name 'quaternionic Maxwell operator' for $\mathcal{N}$. Functions from the $\ker\mathcal{N}$ will sometimes be called 'quaternionic monochromatic functions' and the notation $\mathfrak{N}(\Omega)$ will be used.

We can now establish the connection between the operator $\mathcal{N}$ and the mutually conjugate Cauchy–Riemann operators introduced earlier. Indeed, denoting

$$A_1 := \begin{pmatrix} \alpha & -\sigma \\ -\alpha & -\sigma \end{pmatrix} \qquad B_1 := \frac{1}{2}\begin{pmatrix} \sigma^{-1} & -\sigma^{-1} \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix} \tag{3.4}$$

(both matrices are invertible) we can easily verify the equality

$$A_1 \cdot \mathcal{N} \cdot B_1 = \begin{pmatrix} \overline{D_\alpha} & 0 \\ 0 & D_\alpha \end{pmatrix}. \tag{3.5}$$

Multiplying both sides of (3.5) by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the left-hand side and on the right-hand side we arrive at

$$A_2 \cdot \mathcal{N} \cdot B_2 = \begin{pmatrix} D_\alpha & 0 \\ 0 & \overline{D_\alpha} \end{pmatrix} \tag{3.6}$$

with the invertible matrices

$$A_2 := \begin{pmatrix} -\alpha & -\sigma \\ \alpha & -\sigma \end{pmatrix} \qquad B_2 := \frac{1}{2}\begin{pmatrix} -\sigma^{-1} & \sigma^{-1} \\ \alpha^{-1} & \alpha^{-1} \end{pmatrix}. \tag{3.7}$$

A comparison of (2.5), (3.5) and (3.6) leads to the equality

$$A_1 \cdot \mathcal{N} \cdot B_1 \cdot A_2 \mathcal{N} \cdot B_2 = \begin{pmatrix} \Delta+\lambda & 0 \\ 0 & \Delta+\lambda \end{pmatrix} \tag{3.8}$$

or, equivalently, to the equality

$$(A\mathcal{N}B)^2 = \begin{pmatrix} \Delta+\lambda & 0 \\ 0 & \Delta+\lambda \end{pmatrix} \tag{3.9}$$

with

$$A := A_1 \qquad B := B_1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let E_m denote the unit $(m\times m)$ matrix. Equality (3.8) can be considered as the factorization of the Helmholtz operator $(\Delta+\lambda)\cdot E_2$, acting on $\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C})$-valued functions, into the product of two operators $A_1\mathcal{N}B_1$ and $A_2\mathcal{N}B_2$ each of them being similar to the quaternionic Maxwell operator $\mathcal{N}$. Equality (3.9) shows that the other operator, being similar to $\mathcal{N}$, namely, the operator $A\mathcal{N}B$, is a square root of $(\Delta+\lambda)E_2$.

Theorem 4. (Connection between the sets of quaternionic monochromatic functions and quaternionic hyperholomorphic functions.) Let $\mathfrak{N} := \ker\mathcal{N}$ be the set of all quaternionic monochromatic functions, $\mathfrak{M}_\alpha(\Omega)$ be the set of all left-α-hyperholomorphic functions. Then

$$\begin{aligned}\mathfrak{N} &= B_2\cdot(\mathfrak{M}_\alpha \times \bar{\mathfrak{M}}_\alpha)\\ &= \left\{\left(-\frac{1}{2\sigma}(f-g);\ \frac{1}{2\alpha}(f+g)\right) \mid (f,g)\in\mathcal{M}_\alpha\times\bar{\mathcal{M}}_\alpha\right\}\end{aligned} \tag{3.10}$$

where the invertible matrix B_2 is given by (3.7).

Of course this assertion, being important by its significance, is a very simple corollary of (3.5).

Theorem 5. Let $\mathcal{H}_\lambda^{(2)} := \ker(\Delta+\lambda)E_2$ be the set of all metaharmonic $\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C})$-valued functions. Let

$$q_1 := \frac{1}{2\alpha}A_1\mathcal{N}B_1 \qquad q_2 := \frac{1}{2\alpha}A_2\mathcal{N}B_2 \tag{3.11}$$

and

$$Q_1 := q_1|\mathcal{H}_\lambda^{(2)} \qquad Q_2 := q_2|\mathcal{H}_\lambda^{(2)}. \tag{3.12}$$

Then:

(i) The following correlations are true:

(a) $Q_1^2 = Q_1$; $Q_2^2 = Q_2$;

(b) $Q_1\cdot Q_2 = Q_2\cdot Q_1 = 0$;

(c) $Q_1 + Q_2 = I$;

(d) range $(Q_1) = \ker(\pi_\alpha|C^2)\times\ker(\bar{\pi}_\alpha|c^2) = \tilde{\mathcal{H}}_\alpha\times\mathcal{H}_\alpha$, range $(Q_2) = \ker(\bar{\pi}_\alpha|C^2)\times\ker(\pi_\alpha|C^2) = \mathcal{H}_\alpha\times\tilde{\mathcal{H}}_\alpha$;

(ii) For $\mathcal{H}_{1,\alpha} := Q_1(\mathcal{H}_\lambda^{(2)})$; $\mathcal{H}_{2,\alpha} := Q_2(\mathcal{H}_\lambda^{(2)})$

$$\mathcal{H}_\lambda^{(2)} = \mathcal{H}_{1,\alpha}\oplus\mathcal{H}_{2,\alpha}$$

holds.

(iii) $\mathcal{H}_{1,\alpha} = \ker(q_2|C^2)$; $\mathcal{H}_{2,\alpha} = \ker(q_1|C^2)$;

(iv) $\mathcal{H}_\lambda^{(2)} = \ker(q_2|C^2)\oplus\ker(q_1|C^2) = B_2^{-1}(\ker(\mathcal{N}|C^2))\oplus B_1^{-1}(\ker(\mathcal{N}|C^2))$.

Equality (3.2) defining the quaternionic Maxwell operator contains the left operator D. This results, in particular, in the fact that in (3.5)–(3.6) we have just the left mutually conjugate Cauchy–Riemann operators. Of course we can obtain, completely symmetrically, the 'right-hand side' results. Denote by $\mathcal{N}^{(r)}$ the right quaternionic Maxwell operator:

$$\mathcal{N}^{(r)} := \begin{pmatrix} \sigma & -D^{\mathrm{st}} \\ D^{\mathrm{st}} & -\mathrm{i}\omega\mu \end{pmatrix}.$$

The following equalities are true:

$$A_1\cdot\mathcal{N}^{(r)}\cdot B_1 = \begin{pmatrix} {}_\alpha\bar{D}^{\mathrm{st}} & 0 \\ 0 & {}_\alpha\bar{D}^{\mathrm{st}} \end{pmatrix}$$

$$A_2\cdot\mathcal{N}^{(r)}\cdot B_2 = \begin{pmatrix} {}_\alpha D^{\mathrm{st}} & 0 \\ 0 & {}_\alpha\bar{D}^{\mathrm{st}} \end{pmatrix}$$

where A_1, A_2, B_1, B_2 are from (3.4) and (3.7). It is easy to write the analogues of equalities (3.8)–(3.9) and theorems 4 and 5. In addition to the already introduced multiplications, we shall need below the usual multiplication of matrices. Let us denote this operation by '$\star$'.

Theorem 6. (Variants of the Stokes formula congruent with the notion of the quaternionic Maxwell function.) Let $f = (f^1, f^2)$, $g = (g^1, g^2)$, $\{f, g\}\subset C^1(\bar{\Omega};\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C}))$.

Define $\tilde{\sigma}_\psi := (\sigma_\psi;\sigma_{\bar{\psi}})$. Then

(i) $d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\mathrm{st}}\star(B_2^{-1}\circ f)) = ((A_2\circ\mathcal{N}^{(r)}\circ g)\bullet(B_2^{-1}\circ f) + (B_2^{-1}\circ g)\star(A_2\circ\mathcal{N}\circ f) - 2\alpha g\star f)\,\mathrm{d}x$; in particular, if $h := \begin{pmatrix} 0 \\ \alpha^{-1} \end{pmatrix}$ then substituting subsequently $g := h$ and $f := h$ we get

(a) $d(\tilde{\sigma}_{\mathrm{st}}\star(B_2^{-1}\circ f)) = \left(\alpha\cdot B_2^{-1}\circ f + A_2\circ\mathcal{N}[f] - \begin{pmatrix} 0 \\ 2f^2 \end{pmatrix}\right)\mathrm{d}x$

(b) $d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}) = \left(\alpha\cdot B_2^{-1}\circ g + A_2\circ\mathcal{N}^{(r)}[g] - \begin{pmatrix}0\\2g^2\end{pmatrix}\right)\mathrm{d}x$

(ii) $d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f)) = ((A_2\circ\mathcal{N}^{(r)}\circ g)\star(B_1^{-1}\circ f) - (B_2^{-1}\circ g)\star(A_1\circ\mathcal{N}\circ f))\,\mathrm{d}x$ in particular, we have

(a) $d(\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f)) = (\alpha\cdot B_1^{-1}\circ f - A_1\circ\mathcal{N}[f])\,\mathrm{d}x$

(b) $d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}) = (A_2\circ\mathcal{N}^{(r)}[g] - \alpha\cdot B_2^{-1}\circ g)\,\mathrm{d}x$

(iii) $\int_\Gamma(B^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f) = \int_\Omega((A_2\circ\mathcal{N}^{(r)}[g](x))\star(B_2^{-1}\circ f(x)) + (B_2^{-1}\circ g)\star(A_2\circ\mathcal{N}\circ f) - 2\alpha g\star f)\,\mathrm{d}x$ in particular,

(a) $\int_\Gamma\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f) = \int_\Omega\left(\alpha B_2^{-1}\circ f - \begin{pmatrix}0\\2f^2\end{pmatrix} + A_2\circ\mathcal{N}[f]\right)\mathrm{d}x$

(b) $\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}} = \int_\Omega\left(\alpha\cdot B_2^{-1}\circ g - \begin{pmatrix}0\\2g^2\end{pmatrix} + A_2\circ\mathcal{N}^{(r)}[g]\right)\mathrm{d}x$

(iv) $\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f) = \int_\Omega((A_2\circ\mathcal{N}^{(r)}[g])\star(B_1^{-1}\circ f) - (B_2^{-1}\circ g)\star(A_1\circ\mathcal{N}[f]))\,\mathrm{d}x$ in particular:

(a) $\int_\Gamma\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f) = \int_\Omega(\alpha B_1^{-1}\circ f - A_1\circ\mathcal{N}[f])\,\mathrm{d}x$

(b) $\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}} = \int_\Omega(A_2\circ\mathcal{N}^{(r)}[g] - \alpha B_2^{-1}\circ g)\,\mathrm{d}x$.

Proof. Direct verification is obtained using theorem 1; the equalities (3.5)–(3.6); the fact that any of the multiplication by A_1, A_2, B_1, B_2 operators is an automorphism of $\mathbb{C}^1(\bar{\Omega};\mathbb{H}(\mathbb{C})\times\mathbb{H}(\mathbb{C}))$.

Theorem 7. (Variants of the Cauchy integral theorem for quaternionic Maxwell functions.) Let $g\in\mathfrak{M}^{(r)}(\bar{\Omega})$, $f\in\mathfrak{M}(\bar{\Omega})$. Then

(i)

$$d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f)) = -2\alpha g\star f\,\mathrm{d}x$$
$$d(\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f)) = \left(\alpha\cdot B_2^{-1}\circ f - \begin{pmatrix}0\\2f^2\end{pmatrix}\right)\mathrm{d}x$$
$$d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}) = \left(\alpha\cdot B_2^{-1}\circ g - \begin{pmatrix}0\\2g^2\end{pmatrix}\right)\mathrm{d}x$$

(ii)

$$d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f)) = 0$$
$$d(\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f)) = \alpha\cdot B_1^{-1}\circ f\,\mathrm{d}x$$
$$d((B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}) = -\alpha B_2^{-1}\circ g\,\mathrm{d}x$$

(iii)

$$\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f) = -2\alpha\int_\Omega g(x)\star f(x)\,\mathrm{d}x$$
$$\int_\Gamma\tilde{\sigma}_{\text{st}}\star(B_2^{-1}\circ f) = \int_\Omega\left(\alpha\cdot B_2^{-1}\circ f - \begin{pmatrix}0\\2f^2\end{pmatrix}\right)\mathrm{d}x$$
$$\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}} = \int_\Omega\left(\alpha\cdot B_2^{-1}\circ g - \begin{pmatrix}0\\2g^2\end{pmatrix}\right)\mathrm{d}x$$

(iv)

$$\int_\Gamma(B_2^{-1}\circ g)\star\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f) = 0$$
$$\int_\Gamma\tilde{\sigma}_{\text{st}}\star(B_1^{-1}\circ f) = \alpha\int_\Omega B_1^{-1}\circ f\,\mathrm{d}x$$

$$\int_\Gamma (B_2^{-1} \circ g) \star \tilde{\sigma}_{\mathrm{st}} = -\alpha \int_\Omega B_2^{-1} \circ g \, \mathrm{d}x.$$

Proof. The result is a direct corollary of theorem 5. □

Now we are going to introduce the Cauchy kernel for the quaternionic Maxwell function theory, i.e. the fundamental solution of the operator $\mathcal{N}$ for the 'left' theory and the fundamental solution of the operator $\mathcal{N}^{(r)}$ for the 'right' theory. Introduce the notation:

$$\mathcal{K}_{\mathcal{N},\alpha} := B_1 \circ A_2 \circ \mathcal{N} \circ B_2 \circ A_1 \circ \left[\begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \right]. \tag{3.13}$$

Then (3.8) immediately gives

$$\begin{aligned} \mathcal{N}[\mathcal{K}_{\mathcal{N},\alpha}] &= \mathcal{N} \circ B_1 \circ A_2 \circ \mathcal{N} \circ B_2 \circ A_1 \left[\begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \right] \\ &= A_1^{-1} \circ (\Delta + \lambda) E_2 \circ A_1 \circ \left[\begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \right] = \delta \end{aligned}$$

and so $\mathcal{K}_{\mathcal{N},\alpha}$ is the fundamental solution. There is a direct connection between the quaternionic Maxwell–Cauchy kernel $\mathcal{K}_{\mathcal{N},\alpha}$ and the corresponding Cauchy kernels $\mathcal{K}_\alpha$ and $\bar{\mathcal{K}}_\alpha$ of the hyperholomorphic function theory. We have

$$\begin{aligned} \mathcal{K}_{\mathcal{N},\alpha} &:= B_1 \circ A_2 \circ \mathcal{N} \circ B_2 \circ A_1 \circ \left[\begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \right] \qquad \text{(use (3.6))} \\ &= B_1 \circ \begin{pmatrix} D_\alpha & 0 \\ 0 & \bar{D}_\alpha \end{pmatrix} \circ A_1 \left[\begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \right] \\ &= B_1 \circ \begin{pmatrix} \alpha \cdot D_\alpha; & -\sigma \cdot D_\alpha \\ -\alpha \cdot \bar{D}_\alpha; & -\sigma \cdot \bar{D}_\alpha \end{pmatrix} \circ \begin{pmatrix} \theta_\lambda \\ \theta_\lambda \end{pmatrix} \end{aligned}$$

so that finally

$$\mathcal{K}_{\mathcal{N},\alpha} = B_1 \circ \begin{pmatrix} \alpha - \sigma & 0 \\ 0 & -(\alpha + \sigma) \end{pmatrix} \circ \begin{pmatrix} \bar{\mathcal{K}}_\alpha \\ \mathcal{K}_\alpha \end{pmatrix}. \tag{3.14}$$

Furthermore, all factors in (3.13), besides $\mathcal{N}$, are $\mathbb{C}$-valued, and so nothing will change if we substitute $\mathcal{N}$ by $\mathcal{N}^{(r)}$ which means that $\mathcal{K}_{\mathcal{N},\alpha}$ serves both for the left and for the right theories.

Introduce the analogues of the operators ${}^\psi K_\alpha$ and ${}^\psi T_\alpha$ given earlier:

$$K_{\mathcal{N},\alpha}[f](x) := B_1 \circ \int_\Gamma (C^{-1} \circ \mathcal{K}_{\mathcal{N},\alpha}(x - \tau)) \star \tilde{\sigma}_{st,\tau} \star (B_1^{-1} \circ f)(\tau) \tag{3.15}$$

$$T_{\mathcal{N},\alpha}[f](x) := B_1 \circ \int_\Omega (C^{-1} \mathcal{K}_{\mathcal{N},\alpha}(x - \tau)) \star (A_1 \circ f)(\tau) \, \mathrm{d}\tau. \tag{3.16}$$

Theorem 8. (The Borel–Pompeiu (= Cauchy–Green) formula for quaternionic Maxwell functions.)

Let Ω be a bounded domain in $\mathbb{R}^3$ with the closed Lipaunov boundary $\Gamma = \partial\Omega$; let $f \in C^1(\Omega; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C}))$. Then

$$f(x) = K_{\mathcal{N},\alpha}[f](x) + T_{\mathcal{N},\alpha} \cdot \mathcal{N}[f](x) \tag{3.17}$$

for $\forall x \in \Omega$.

Theorem 9. (Right inverse to the quaternionic Maxwell operator.) Let $f \in C^{0,\mu}(\Omega) \cap C(\bar{\Omega})$, $0 < \mu \leqslant 1$. Then in Ω the equality

$$\mathcal{N} \cdot T_{\mathcal{N},\alpha}[f](x) = f(x) \tag{3.18}$$

holds.

Theorem 10. (The Cauchy integral formula for quaternionic Maxwell functions.) Let $f \in \mathcal{N}(\Omega; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}; \mathbb{H}(\mathbb{C}) \times \mathbb{H}(\mathbb{C}))$. Then

$$f(x) = K_{\mathcal{N},\alpha}[f](x) \tag{3.19}$$

for $\forall x \in \Omega$.

Having in mind the analogues of the Plemelj–Sokhotski formulae, the Morera theorem and other properties of (ψ, α)-hyperholomorphic functions [25, 26, 2], the corresponding theorems for quaternionic monochromatic functions can easily be obtained. Then, of course, we can obtain them for the usual time-harmonic Maxwell equations as a restriction of the above results onto the three-dimensional case (see [20, 23, 26]). It should be noted as well that (ψ, α)-hyperholomorphic function theory is closely related to the theory of spinor fields (see [22, 26]) which leads to a natural connection between electromagnetic and spinor fields.

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